OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 4 NOVEMBER 2009

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

<u>NOTE</u>: **Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.**

NAME:
TEST CENTRE:
OXFORD COLLEGE (if known):
DEGREE COURSE:
DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ____________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The smallest value of

$$I(a) = \int_0^1 \left(x^2 - a\right)^2 \, \mathrm{d}x,$$

as a varies, is

(a) $\frac{3}{20}$, (b) $\frac{4}{45}$, (c) $\frac{7}{13}$, (d) 1.

B. The point on the circle

$$x^2 + y^2 + 6x + 8y = 75,$$

which is closest to the origin, is at what distance from the origin?

(a) 3, (b) 4, (c) 5, (d) 10.

C. Given a real constant c, the equation

$$x^4 = (x-c)^2$$

has four real solutions (including possible repeated roots) for

(a) $c \leqslant \frac{1}{4}$, (b) $-\frac{1}{4} \leqslant c \leqslant \frac{1}{4}$, (c) $c \leqslant -\frac{1}{4}$, (d) all values of c.

D. The smallest positive integer n such that

$$1-2+3-4+5-6+\cdots+(-1)^{n+1}\,n \geqslant 100,$$

is

(a) 99, (b) 101, (c) 199, (d) 300.

E. In the range $0 \leqslant x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

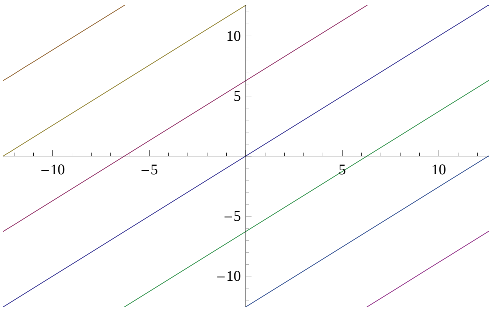
(a) has 0 solutions;
(b) has 1 solution;
(c) has 2 solutions;
(d) holds for all values of x.

F. The equation in x

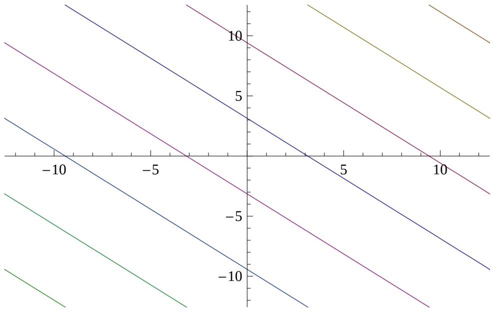
$$3x^4 - 16x^3 + 18x^2 + k = 0$$

has four real solutions
(a) when $-27 < k < 5$;
(b) when $5 < k < 27$;
(c) when $-27 < k < -5$;
(d) when $-5 < k < 0$.

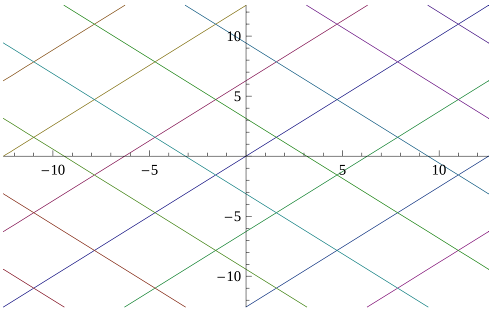
G. The graph of all those points (x, y) in the xy-plane which satisfy the equation $\sin y = \sin x$ is drawn in



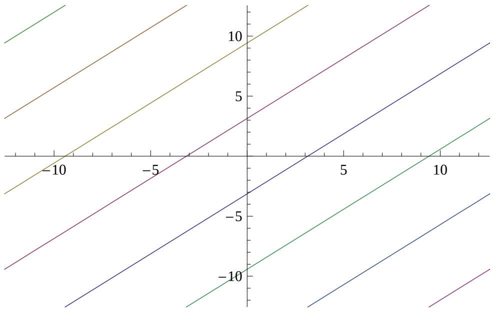
(a)



(b)



(c)



(d)

H. When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x \, \mathrm{d}x$$

by dividing the interval $0 \leqslant x \leqslant 1$ into N subintervals the answer achieved is

(a) $\dfrac{1}{2N}\left\{1 + \dfrac{1}{2^{1/N}+1}\right\}$, (b) $\dfrac{1}{2N}\left\{1 + \dfrac{2}{2^{1/N}-1}\right\}$,

(c) $\dfrac{1}{N}\left\{1 - \dfrac{1}{(2^{1/N}-1)}\right\}$, (d) $\dfrac{1}{2N}\left\{\dfrac{5}{2^{1/N}+1} - 1\right\}$.

I. The polynomial

$$n^2 x^{2n+3} - 25n x^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor
(a) for no values of n;
(b) for $n = 10$ only;
(c) for $n = 15$ only;
(d) for $n = 10$ and $n = 15$ only.

J. The number of *pairs* of *positive integers* x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(a) 0, (b) 2^6, (c) $2^9 - 1$, (d) $2^{10} + 2$.

2. For **ALL APPLICANTS.**

A list of real numbers $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 1$, $x_2 = 3$ and then for $n \geqslant 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of x_4 and x_5.

(ii) Find values of real constants A, B, C such that <u>for $n = 1, 2, 3$</u>,

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true <u>for all $n \geqslant 1$</u>, find the smallest n such that $x_n \geqslant 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \ldots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n$$

Find, explaining your reasoning, a formula for y_n which holds for $n \geqslant 2$.

What is the approximate value of x_n/y_n for large values of n?

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

For a positive whole number n, the function $f_n(x)$ is defined by

$$f_n(x) = \left(x^{2n-1} - 1\right)^2.$$

(i) On the axes provided opposite, sketch the graph of $y = f_2(x)$ labelling where the graph meets the axes.

(ii) On the same axes sketch the graph of $y = f_n(x)$ where n is a large positive integer.

(iii) Determine

$$\int_0^1 f_n(x)\,\mathrm{d}x.$$

(iv) The *positive* constants A and B are such that

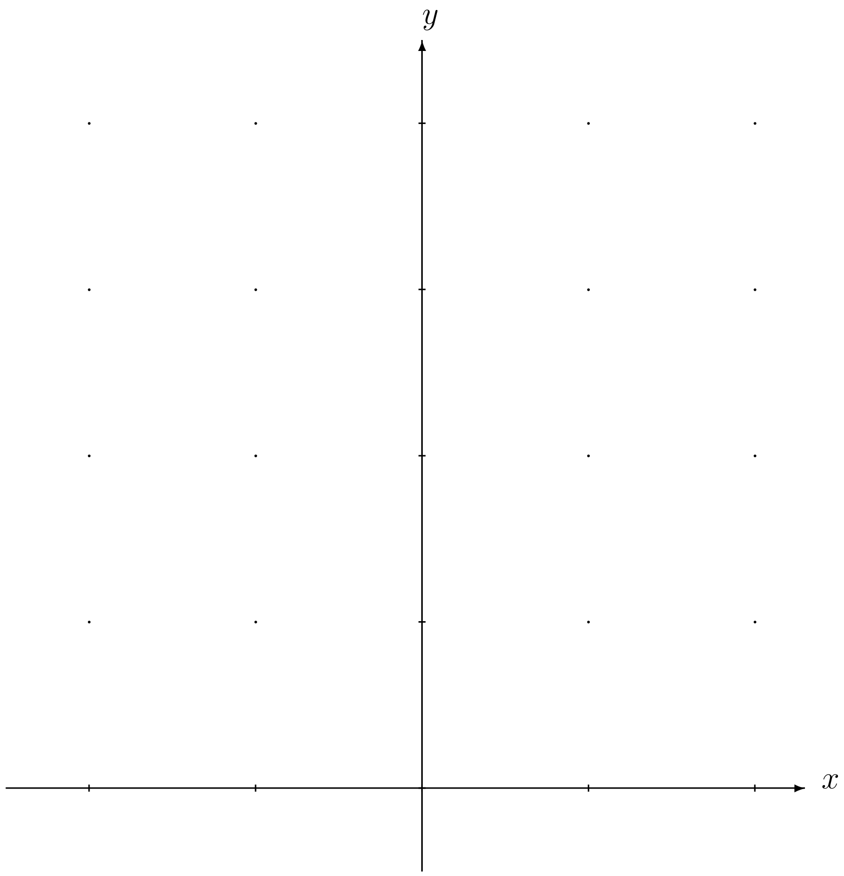
$$\int_0^1 f_n(x)\,\mathrm{d}x \leqslant 1 - \frac{A}{n+B} \quad \text{for all } n \geqslant 1.$$

Show that

$$(3n-1)(n+B) \geqslant A(4n-1)n,$$

and explain why $A \leqslant 3/4$.

(v) When $A = 3/4$, what is the smallest possible value of B?

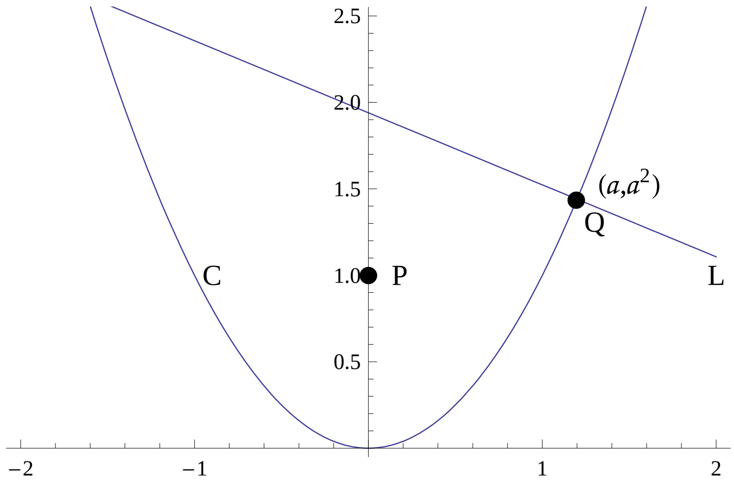

y
x

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**

As shown in the diagram below: C is the parabola with equation $y = x^2$; P is the point $(0, 1)$; Q is the point (a, a^2) on C; L is the normal to C which passes through Q.



(i) Find the equation of L.

(ii) For what values of a does L pass through P?

(iii) Determine $|QP|^2$ as a function of a, where $|QP|$ denotes the distance from P to Q.

(iv) Find the values of a for which $|QP|$ is smallest.

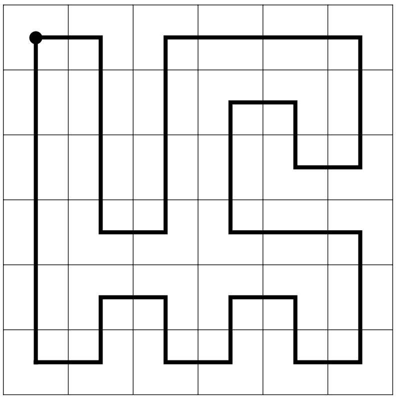
(v) Find a point R, in the xy-plane but not on C, such that $|RQ|$ is smallest for a unique value of a. Briefly justify your answer.

5. For **ALL APPLICANTS.**

Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within the grid such that:

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre of an adjacent square;
- the path starts and finishes in the same square;
- the path visits the centre of every other square just once.

For example, below is a tour drawn in a 6×6 grid of squares which starts and finishes in the top-left square.



For parts (i)-(iv) it is assumed that n is *even*.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid.

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer.

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands two commands:

- command R which turns the robot clockwise through a right angle;
- command F which moves the robot forward to the centre of the next square.

The robot has a program, a list of commands, which it performs in the given order to complete a tour; say that, in total, command R appears r times in the program and command F appears f times.

(iii) Initially the robot is in the top-left square pointing to the right. Assuming the first command is an F, what is the value of f? Explain also why $r + 1$ is a multiple of 4.

(iv) Must the results of part (iii) still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning.

(v) Show that a tour of an $n \times n$ grid is not possible when n is odd.

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

(i) Alice, Bob, and Charlie make the following statements:

Alice:	Bob is lying.
Bob:	Charlie is lying.
Charlie:	$1 + 1 = 2$.

Who is telling the truth? Who is lying?. Explain your answer.

(ii) Now Alice, Bob, and Charlie make the following statements:

Alice:	Bob is telling the truth.
Bob:	Alice is telling the truth.
Charlie:	Alice is lying.

What are the possible numbers of people telling the truth? Explain your answer.

(iii) They now make the following statements:

Alice:	Bob and Charlie are both lying.
Bob:	Alice is telling the truth or Charlie is lying (or both).
Charlie:	Alice and Bob are both telling the truth.

Who is telling the truth and who is lying on this occasion? Explain your answer.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length n. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length n that start with an M, an X, a W respectively.

Explain why

$$\begin{aligned} m(n) &= w(n), \\ m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\ x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1, \end{aligned}$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length n.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that

$$r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}, \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even}. \end{cases}$$

End of Last Question